Homogeneous random fields and statistical mechanics *

by

Sergio Albeverio and Raphael Høegh-Krohn

Institute of Mathematics
University of Oslo
Blindern, Oslo (Norway)

A B S T R A C T

We illustrate the connection between homogeneous perturbations of homogeneous Gaussian random fields over R^n or Z^n, with values in R^m, and classical as well as quantum statistical mechanics. In particular we construct homogeneous non Gaussian random fields as weak limits of perturbed Gaussian random fields and study the infinite volume limit of correlation functions for a classical continuous gas of particles with inner degrees of freedom. We also exhibit the relation between quantum statistical mechanics of lattice systems (anharmonic crystals) at temperature β^{-1} and homogeneous random fields over $Z^n \times S_\beta$, where S_β is the circle of length β, which then provides a connection also with classical statistical mechanics. We obtain the infinite volume limit of real and imaginary times Green's functions and establish its properties. We also give similar results for the Gibbs state of the correspondent classical lattice systems and show that it is the limit as $h \to 0$ of the quantum statistical Gibbs state.

May 8, 1974.

* Work supported by The Norwegian Research Council for Science and the Humanities.

1. Introduction

The main purpose of this paper is to illustrate the intimate connection which exists between the study of homogeneous perturbations of homogeneous Gaussian random fields and the study ot the basic quantities of classical and quantum non relativistic statistical mechanics.

This connection complements the one discussed in recent years which relates the study of homogeneous perturbations of homogeneous Gaussian generalized Markoff random fields to the study of constructive quantum field theory (see e.g. [1], [2], [3]) and relativistic quantum statistical mechanics (quantum field theory at non zero temperature) [4].

Random fields also play a unifying role from another point of view, in that they are the basic quantities of stochastic mechanics [5] and stochastic field theory [6], which in turn are closely related to non relativistic quantum mechanics [5], the theory of the heat equation [7] and euclidean quantum field theory [6]. The idea of the relevance of stochastic processes and functional integration in the study of all the mentioned subjects can historically be traced back particularly to studies developed in connection on one hand with the Brownian motion (see e.g. [5]) and on the other hand with Feynman's formulation of quantum mechanics and quantumelectrodynamics in terms of path space integrals (see e.g. [8], [9], [10]). In particular the relation with the Wiener integral (see e.g. [11], [12]) has found applications in non relativistic quantum statistical mechanics, particularly through the work of Ginibre on reduced density matrices for dilute non relativistic gases in thermal equilibrium (see e.g. [13]; also [14], [15]). In this work the reduced density matrices are expressed in terms of correlation functions of a classical gas, for which the well known results of Ruelle [16] apply. The method used for this is essentially Feynman-Kac formula (see e.g. [12]) to express the statistic operator by an integral of a numerical function over Wiener trajectorics.

In quantum field theory foundational work on integration with respect to Gaussian generalized random fields has been done quite early, particularly by Friedrichs [17] and Segal [18]. Symanzik [19] stressed the connection, relating also to Ginibre's work, between the study of Euclidean quantum field models and both classical and quantum statistical mechanics. Symanzik's program could be realized in two-dimensional models, only after fundamental work in constructive quantum field theory, carried through particularly by J. Glimm and A. Jaffe, and after Nelson's introduction of Euclidean Markoff methods ([20],[1]). First applications were [21] and, exploiting the relation with statistical mechanics, [2],[22a]. [1)] These results can be looked upon as constructions of homogeneous non Gaussian generalized random fields over R^2, as weak limits of non homogeneous non Gaussian generalized Markoff random fields, attached to finite regions of R^2. Results on support properties of the free measure ([1],[23]) and on the Markoff property in the limit are also known [24].

One of us [4] has extended the work on quantum field models from the zero temperature case to the case of positive temperature, constructing the unique infinite volume Green's functions and Gibbs state for relativistic quantum statistical models in two-dimensional space-time. The method uses a representation of the state in terms of expectations with respect to a homogeneous Gaussian generalized random field on $S_\beta \times R$, where S_β is the circle of length β. The state was proven to be translation invariant, KMS, strongly mixing and analytic properties of the Green's functions were established as well as a duality principle, asserting the equality of the imaginary time Green's functions at temperature $1/\beta$ with those with space and time interchanged, of a correspondent system at temperature zero in a periodic box of length β .

Before going over to the description of the containt of the present paper, let us remark that for certain classical lattice systems an equivalence between the descriptions in terms of the Gibbs ensemble and in terms of Markovian random fields is known (see e.g. [25]).

In the present paper, ideas of [2], [3] and [4] are joined together and applied to the study of classical continuous and discrete statistical systems, to lattice quantum statistical systems with continuous degrees of freedom as well as to the study of homogeneous perturbations of homogeneous Gaussian random fields.

In section 2 we study homogeneous perturbations of homogeneous Gaussian random fields $\xi(x)$ over R^n (or Z^n), with values in R^n and with bounded covariance matrix valued function $G(x-y)$.

Starting with the underlying probability measure dP_0 , we define, for any bounded domain Λ of R^n , a new measure

$dP_\Lambda = (\int e^{\lambda\int_\Lambda f(\xi(x))dx} dP_0)^{-1} e^{\lambda\int_\Lambda f(\xi(x))dx} dP_0$, where $f(\cdot)$ is any function which can be written as Fouriertransform of some complex finite measure $d\mu$ on R^m , decreasing at infinity. We then show that dP_Λ converges weakly as $\Lambda \to R^n$ to the measure dP for a non gaussian homogeneous random field, provided $|\lambda| < \lambda_0$, $\lambda_0 > 0$. We call perturbations of the above form "gentle". The characteristic functional of dP is analytic for $|\lambda| < \lambda_0$ and we give its power series (linked cluster) expansion. Moreover dP is strongly

mixing. These results are closely related to the ones developed for a class of quantum field theoretical models in [2] and their proof uses the fact that the Fouriertransforms $\int e^{i\Sigma\alpha_j \xi(x_j)} dP$ are essentially the correlation functions for a classical gas of particles which have, in addition to the usual translational degrees of freedom, also a discrete or continuous m-dimensional degree of freedom α, distributed with weight proportional to $d\mu(\alpha)$. These particles interact by two-body potentials $\alpha_i G_{ij}(x_i - x_j)\alpha_j$, where α_i is the non translational degree of freedom of particle i.
An example is provided by the case where the particles are orientable (e.g. diatomic) molecules, with α_i a unit vector giving the orientation of the i-th molecule and $d\mu(\alpha)$ is a bounded measure on the unit sphere. Another example is $d\mu(\alpha) = \delta_{\frac{1}{\beta^{\frac{1}{2}}}}(\alpha)$, in which case we have simply the usual classical gas of particles with temperature $1/\beta$ and activity λ, interacting by two-body potentials. A third example is the case $m = 1$, $d\mu(\alpha) = \delta(\alpha-1) + \delta(\alpha+1)$, in which we have a gas of scalar spin particles interacting by two-body potentials $\alpha_i G_{ij}(x_i - x_j)\alpha_j$ with $\alpha_i, \alpha_j = \pm 1$.
The proof of the convergence of the correlation functions for $|\lambda| < \lambda_0$ and of their linked cluster expansion is modelled after the method of Kirkwood-Salzburg equations ([16],[26].

For $|\lambda| \geq \lambda_0$ we have only weak convergence through subsequences for the measure dP_Λ and the correlation functions, a result obtained for other types of interactions by Dobrushin [27] and Ruelle [28]. The case of perturbations given in terms of unbounded functions $f(\cdot)$ is also shortly discussed in the discrete case, using correlation inequalities along similar lines as in [3]. This then establishes the connection with well known work on ferromagnetic systems (see e.g. [29].

In section 3 we represent as in [4] the Gibbs state and Green's functions of the N-dimensional quantum mechanical anharmonic oscillator by expectations with respect to the homogeneous Gaussian process on the circle of length β and covariance given in terms of the harmonic part of the potential. This representation is the basis for the connection of quantum statistical quantities with classical statistical ones. In section 4, following lines of [4], we exhibit and apply this connection to
the study of d-dimensional statistical mechanical quantum lattice systems, with m continuous degrees of freedom, associated with each lattice site n, and finite volume Hamiltonian of the form

$$H(\Lambda) = -\tfrac{1}{2}\sum_{n\in\Lambda}\Delta_n + \sum_{n,n'\in\Lambda} x_n A(n-n')x_{n'} + \lambda\sum_{n\in\Lambda} f(x_{n+a_1},\dots,x_{n+a_k}), \quad (1)$$

where Λ is a finite subset of the lattice Z^d, Δ_n is the Laplacian with respect to x_n, $A(n)$ is a matrix-valued function satisfying finite range and stability conditions (see (4.2)), $f(\cdot)$ is any function as above and $a_1,\dots,a_k$ are fixed lattice vectors. We can e.g. look upon (1) as the Hamiltonian for a anharmonic quantum mechanical crystal (cfr. e.g. [30]) and in this case we interpret x_n as the deplacement of a particle associated with the lattice site n. This particle is bound to n by the positive harmonic potential $\tfrac{1}{2}x_n(0)x_n$ and interacts with its neighbors within a fixed distance by the harmonic potential $\tfrac{1}{2}\sum_{n'} x_n A(n-n')x_n$. Moreover the same particle interacts with $(k-1)$ other particles associated will the lattice points $n+(a_2-a_1),\dots,n+(a_k-a_1)$, where k and $a_1,\dots,a_k$ are independent of the lattice site n, by the anharmonic potential $\lambda f(x_n, x_{n+(a_2-a_1)},\dots,x_{n+(a_k-a_1)})$. We give an expression for the finite volume Green functions for the system at temperature β and covariance given in terms of the harmonic terms in (1). This reduces then the problem of the infinite volume limit to the one of homogeneous gentle perturbations of homogeneous random fields, solved in Section 2 by exhibiting its connection with classical statistical mechanics. We prove convergence of real time and imaginary times Green's functions as $\Lambda \to R^n$ (Z^n) and $|\lambda| < \lambda_0$ and exhibit analyticity domains in λ and the time difference variables as well as simple uniform bounds. We also have convergent linked cluster expansions. The corresponding Gibbs state is a translation invariant KMS state. These results complement those obtained for quantum field theoretical [1],[2] and relativistic quantum statistical systems [4], as well as for lattice quantum spin systems (e.g. [30],[31]), dilute non relativistic Fermi gases ([32]), and dilute non relativistic Bose gases ([33]).

In section 5 we establish the analogous results for the Gibbs state of the correspondent classical lattice systems with harmonic and anharmonic interactions (e.g. classical anharmonic crystals). Again the method used is to reduce the problem to the homogeneous gentle perturbations of a Gaussian random field discussed in Section 2. We prove also that the equal time Green's functions for the quantum mechanical system with finite volume Hamiltonian (1), with Δ_n replaced by $\hbar^{-2}\Delta_n$, converge as the Planck's constant h tends to zero to the correspondent correlation functions for the analogue classical system. This is a contribution to the discussion of the problem of "classical limit" (see e.g. [35]).

Applications of the methods and results of the present work to the discussion of phase transitions will be given in a forthcoming paper.

2. Gentle homogeneous perturbations of homogeneous Gaussian random fields.

A random field over Z^n or R^n is a family of random variables $\xi(x)$, parametrized by $x \in Z^n$ or $x \in R^n$, with values in R^m. That is, $\xi(x)$ is for each x a measurable function from a probability space $(\Omega, \mathcal{B}, P)$, where $\mathcal{B}$ is the set of measurable sets in Ω and P is a probability measure defined on $\mathcal{B}$, such that $\xi(x)(\omega)$ is measurable in x and ω with values in R^m. A homogeneous random field is a random field such that for any k the joint distribution of $\xi(x_1),\dots,\xi(x_k)$ is translation invariant, i.e. $\xi(x_1),\dots,\xi(x_k)$ and $\xi(x_1+a),\dots,\xi(x_k+a)$ have the same joint probability distribution for any $a \in Z^n$ or $a \in R^n$. A homogeneous Gaussian field is a homogeneous random field where the joint probability distribution of $\xi(x_1,\dots,\xi(x_k)$ is Gaussian for any k and any $x_1,\dots,x_k$.

In this section we shall study some homogeneous random fields which are close to homogeneous Gaussian fields. Since a joint Gaussian probability distribution is characterized by its expectation and its covariance matrix, we have that a homogeneous Gaussian field is characterized by its expectation $E(\xi_i(x)) = m_i$ and its correlation function

$$E(\xi_i(x)\xi_j(y) = G_{ij}(x-y) , \tag{2.1}$$

where $\xi_i(x)$ is the i-th component in R^m of $\xi(x)$. Since the field is homogeneous the expectation is independent of x and the covariance depends only on the difference $x-y$.

It follows from (2.1) that, for any $\lambda \in R^m$, $\sum_{i,j} \lambda_i\lambda_j G_{ij}(x)$ is a bounded positive definite function on Z^n or R^n. Since a bounded positive definite function on R^n is continuous, we get that $G_{ij}(x)$ are continuous functions. Moreover we get from (2.1) that for any finite sequence of vectors $\lambda^1,\dots,\lambda^r$ in R^m and $x_1,\dots,x_n$ in R^n

$$\sum_{k,l}\sum_{i,j}\lambda_i^k\lambda_j^l G_{ij}(x_k-x_l) = \sum_{k,l}\lambda^k G(x_k-x_l)\lambda^l \geq 0 \qquad (2.2)$$

A generalized random field over R^n is a linear mapping $\xi(\varphi)$ from the space of smooth functions $\varphi \in C_0^\infty(R^n)$ into the linear space of R^m valued random variables. If $\xi(x)$ is a random field over R^n, then $\xi(x)$ defines in a natural way a generalized random field by the formula $\xi(\varphi) = \int \xi(x)\varphi(x)dx$.

Let now $\xi(x)$ be a homogeneous Gaussian field over R^n or Z^n with $E(\xi(x)) = 0$. We shall discuss a method to construct homogeneous non Gaussian random fields with the help of the Gaussian field $\xi(x)$. Let $(\Omega, \mathcal{B}, P_0)$ be the underlying probability space for the homogeneous Gaussian field $\xi(x)$. Let $f(s_1,\dots,s_k)$, be a bounded continuous real[2] function of k variables $s_1,\dots,s_k$ in R^m. Let $a_1,\dots,a_k$ be vectors in R^n or Z^n, and let P_Λ be the probability measure given by

$$dP_\Lambda = Z_\Lambda^{-1} e^{\lambda\int_\Lambda f(\xi(x+a_1),\dots,\xi(x+a_k))dx} dP_0 , \qquad (2.3)^*$$

where $\lambda \in R$, Λ a bounded domain in R^n or Z^n and

$$Z_\Lambda = \int e^{\lambda\int_\Lambda f(\xi(x+a_1),\dots,\xi(x+a_k))dx} dP_0 , \qquad (2.4)^*$$

with the convention that if $\xi(x)$ is a homogeneous Gaussian field over Z^n then $\int_\Lambda f(\xi(x)dx$ is defined as $\sum_{x\in\Lambda} f(\xi(x))$.

Throughout this section we shall use **more generally** the convention that if x is a variable that runs over Z^n then $\int_\Lambda \cdot\, dx$ is defined as $\sum_{x\in\Lambda}$.

It is obvious that if $\xi(x)$ is homogeneous Gaussian random field with values in R^m, then $\hat{\xi}(x) = \{\xi(x+a_1), \xi(x+a_2), \ldots, \xi(x+a_k)\}$ is a homogeneous Gaussian random field with values in $R^{m\cdot k}$. f may be considered a real function on $R^{m\cdot k}$ and the formulae above may be written:

$$dP_\Lambda = Z_\Lambda^{-1} e^{\lambda \int_\Lambda f(\hat{\xi}(x))dx} d\hat{P}_o \tag{2.3}$$

and

$$Z_\Lambda = \int e^{\lambda \int_\Lambda f(\hat{\xi}(x))dx} d\hat{P}_o , \tag{2.4}$$

where $d\hat{P}_o$ is the probability measure for the homogeneous Gaussian random field $\hat{\xi}(x)$. Since m is arbitrary it is therefore enough to consider the situation $k = 1$, $a_1 = 0$, and we shall therefore write the proofs of the theorems only for the case $k = 1$, $a_1 = 0$.

From (2.3) we see that P_Λ is absolutely continuous with respect to the probability measure P_o. Hence $\xi(x)(\omega)$ are measurable functions defined on the probability space $(\Omega, \mathcal{B}, P_\Lambda)$, and let us denote the corresponding random variables by $\xi_\lambda^\Lambda(x)$. $\xi_\lambda^\Lambda(x)$ is then obviously a random field which is not homogeneous. We have in fact that

$$\xi_\lambda^\Lambda(x-a) = \xi_\lambda^{\Lambda+a}(x) . \tag{2.5}$$

If we could prove that the limit as $\Lambda \to R^n$ of $\xi_\lambda^\Lambda(x)$ exists, then it would follow from (2.5) that the limit would be a homogeneous random field. For this reason let us compute the Fourier transform of the joint probability distribution of $\xi_\lambda^\Lambda(x_1),\ldots,\xi_\lambda^\Lambda(x_k)$. With $\alpha_1,\ldots,\alpha_k$ in R^m we have

$$\varphi_\lambda^\Lambda(x_1\alpha_1,\ldots,x_k\alpha_k) = E\left[e^{i\sum_{i=1}^k \alpha_j\cdot\xi_\Lambda(x_j)}\right] \tag{2.6}$$

$$= Z_\Lambda^{-1}\int e^{i\sum_{j=1}^k \alpha_j\xi(x_j)}\, e^{\lambda\int_\Lambda f(\xi(x))dx}\, dP_o .$$

We shall assume that $f(s)$ is the Fourier transform of a bounded complex measure

$$f(s) = \int e^{is\alpha}\, d\mu(\alpha) , \tag{2.7}$$

where $\mu(\alpha) = \bar{\mu}(-\alpha)$ [3])

with $\|\mu\| = \int d|\mu(\alpha)| < \infty$. Since $f(s)$ is a bounded function we have that

$$\int e^{i\sum_{j=1}^k \alpha_j\xi(x_j)}\, e^{\lambda\int_\Lambda f(\xi(x))dx}\, dP_o$$

$$= \sum_{n=o}^{\infty}\frac{\lambda^n}{n!}\int_{\Lambda^n}\cdots\int\left[\int e^{i\sum_{j=1}^k \alpha_j\xi(x_j)} f(\xi(x_{k+1}))\ldots f(\xi(x_{k+n}))dP_o\right]\prod_{j=k+1}^{k+n} dx_j$$

$$= \sum_{n=o}^{\infty} \frac{\lambda^n}{n!} \int_{\Lambda^n} \cdots \int \Big[\int e^{i \sum_{j=1}^{k+n} \alpha_j \xi(x_j)} dP_o \Big] \prod_{j=k+1}^{k+n} d\mu(\alpha_j) dx_j$$

$$= \sum_{n=o}^{\infty} \frac{\lambda^n}{n!} \int_{\Lambda^n} \cdots \int e^{-\frac{1}{2} \sum_{i,j=1}^{k+n} \alpha_i G(x_i - x_j) \alpha_j} \prod_{j=k+1}^{k+n} d\mu(\alpha_j) dx_j .$$

Hence

$$\varphi_\lambda^\Lambda(x_1\alpha_1, \ldots, x_k\alpha_k) = Z_\Lambda^{-1} \sum_{n=o}^{\infty} \frac{\lambda^n}{n!} \int_{\Lambda^n} \cdots \int e^{-\frac{1}{2} \sum_{i,j=1}^{k+n} \alpha_i G(x_i - x_j) \alpha_j} \prod_{j=k+1}^{k+n} d\mu(\alpha_j) dx_j, \tag{2.8}$$

with

$$Z_\Lambda = \sum_{n=o}^{\infty} \frac{\lambda^n}{n!} \int_{\Lambda^n} \cdots \int e^{-\frac{1}{2} \sum_{i,j=1}^{n} \alpha_i G(x_i - x_j) \alpha_j} \prod_{j=1}^{n} d\mu(\alpha_j) dx_j . \tag{2.9}$$

We may also write (2.8) and (2.9) in the form

$$\varphi_\lambda^\Lambda(x_1\alpha_1, \ldots, x_k\alpha_k) = \lambda^{-k} e^{-\frac{1}{2} \sum_{j=1}^{k} \alpha_j G(o) \alpha_j} \cdot Z_\Lambda^{-1} \sum_{n=o}^{\infty} \frac{\lambda^{n+k}}{n!} \int_{\Lambda^n} \cdots \int e^{-\sum_{i<j}^{n+k} \alpha_i G(x_i - x_j) \alpha_j} \prod_{j=k+1}^{k+n} d\nu(\alpha_j) dx_j , \tag{2.10}$$

with $d\nu(\alpha) = e^{-\frac{1}{2}\alpha G(o) \alpha} d\mu(\alpha)$, and

$$Z_\Lambda = \sum_{n=o}^{\infty} \frac{\lambda^n}{n!} \int_{\Lambda^n} \cdots \int e^{-\sum_{i<j}^{n} \alpha_i G(x_i - x_j) \alpha_j} \prod_{j=1}^{n} d\nu(\alpha_j) dx_j . \tag{2.11}$$

This shows that

$$\varphi_\lambda^\Lambda(x_1\alpha_1, \ldots, x_k\alpha_k) = \lambda^{-k} e^{-\frac{1}{2} \sum_{j=1}^{k} \alpha_j G(o) \alpha_j} \rho_\lambda^\Lambda(x_1\alpha_1, \ldots, x_k\alpha_k), \tag{2.12}$$

where $\rho_\lambda^\Lambda(x_1\alpha_1, \ldots, x_k\alpha_k)$ are the correlation functions for a classical system of interacting particles contained in the bounded domain $\Lambda \subset R^n$, with potential energy given by a twobody interaction of the form

$$U = \sum_{i<j} \alpha_i G(x_i - x_j)\alpha_j . \tag{2.13}$$

The α's correspond to an internal freedom $\alpha \in R^m$, for each particle, and the measure $d\nu(\alpha)$ gives the range of variation for this internal degree of freedom. For $m = 1$ and $d\nu(\alpha) = \delta_{\beta^{\frac{1}{2}}}(\alpha)$ [4] we get that the ρ_λ^Λ are the correlation functions for a conventional classical system of interacting particles at temperature $\frac{1}{\beta}$ and activity λ.

Using standard methods of classical statistical mechanics, namely the Kirkwood-Salzburg equations [see Ref [16] and Ref [2a]] we get that if $\mu(\alpha)$ has support in a sphere of radius r and

$$C = \sup_\alpha \int \left| e^{-\alpha G(x)\beta} - 1 \right| d|\nu|(\beta)dx , \tag{2.14}$$

is finite, than the correlation function $\rho_\lambda^\Lambda(x_1\alpha_1, \ldots, x_k\alpha_k)$ converges uniformly on compact subsets to a limit $\rho_\lambda(x_1\alpha_1, \ldots, x_k\alpha_k)$ as $\Lambda \to R^n$, if $|\lambda| < \lambda_o$, with $\lambda_o = C^{-1} e^{-2B-1}$ and

$$B = \|G(o)\| r^2 , \tag{2.15}$$

where $\|G(o)\|$ is the norm of the matrix $G(o)$. i.e. the largest eigenvalue of the positive definite matrix $G(o)$. Moreover the limit $\rho_\lambda(x_1\alpha_1, \ldots, x_k\alpha_k)$ is translation invariant in x and analytic in λ for λ in the complex disc $|\lambda| < \lambda_o$.

By the relation (2.12) between φ_λ^Λ and ρ_λ^Λ we therefore get that as $\Lambda \to R^n$ [5] in the sense that, for any x, $d(x,\partial\Lambda)$, the distance from x to $\partial\Lambda$, tends to infinity, the limit of $\varphi_\lambda^\Lambda(x_1\alpha_1, \ldots, x_k\alpha_k)$ exists and the convergence is uniform on bounded sets. Moreover the limit $\varphi_\lambda(x_1, \ldots, x_k\alpha_k)$ is translation invariant in x and analytic in λ for $|\lambda| < \lambda_o$. Recalling that $\varphi_\lambda^\Lambda(x_1\alpha_1, \ldots, x_k\alpha_k)$ is the

Fouriertransform of the joint probability distribution of $\xi_\lambda^\Lambda(x_1),\ldots,\xi_\lambda^\Lambda(x_k)$, we see that $\xi_\lambda^\Lambda(x)$ converges to a homogeneous random field $\xi_\lambda(x)$ as $\Lambda \to R^n$, in the sense that the Fouriertransforms of the joint distributions of $\xi_\lambda^\Lambda(x_1),\ldots,\xi_\lambda^\Lambda(x_k)$ converge. We summarize this in a theorem.

Theorem 2.1.

Let $\xi(x)$ be a homogeneous Gaussian field over R^n or Z^n with values in R^m, expectation zero and covariance $E(\xi(x)\xi(y)) = G(x-y)$. Let $f(s_1,\ldots,s_k = \int_{R^{m\cdot k}} e^{i\sum_{j=1}^{k} s_j\alpha_j} d\mu(\alpha_1,\ldots,\alpha_j)$ be a real function, where $\|\mu\| = \int d|\mu|(\alpha) < \infty$ and the support of $\mu(\alpha)$ is contained in a sphere of radius r in $R^{m\cdot k}$. For Λ a bounded region in R^n we define a new random field $\xi_\lambda^\Lambda(x)$, where $\xi_\lambda^\Lambda(x)$ is the random variable given by the measurable function $\xi(x)(\omega)$ and the probability measure $dP_\lambda^\Lambda = Z_\Lambda^{-1} e^{\lambda\int_\Lambda f(\xi(x+a_1),\ldots,\xi(x+a_k))dx} dP_o$, with $Z_\Lambda = \int e^{\lambda\int_\Lambda f\,dx} dP_o$ and P_o is the underlying probability measure for the homogeneous Gaussian field $\xi(x)$.

If $B = \|G(o)\|r^2$ and

$$C = \sup_{|\alpha|\leq r} \int |e^{-\alpha G(x)\beta}-1|e^{-\frac{1}{2}\beta G(o)\beta} d|\mu(\beta)|dx$$

is finite, then as $\Lambda \to R^n$ in the sense that the distance from $\partial\Lambda$ to any fixed point tends to infinity, the random field $\xi_\lambda^\Lambda(x)$ converges for $|\lambda| < \lambda_o = C^{-1}e^{-2B-1}$ to a homogeneous random field $\xi_\lambda(x)$ which is analytic in λ for λ in the complex disc $|\lambda| < \lambda_o$. The convergence of the random field is in the sense that the Fourier transforms of the joint distributions of $\xi_\lambda^\Lambda(x_1),\ldots,\xi_\lambda^\Lambda(x_k)$ converge for all $x_1,\ldots,x_k$ and uniformly for $x_1,\ldots,x_k$ in bounded sets. I.e.

$$\varphi_\lambda^\Lambda(x_1\alpha_1,\ldots,x_k\alpha_k) = E(e^{i\sum_{j=1}^{k}\alpha_j\xi_\lambda^\Lambda(x_j)})$$

converges uniformly on bounded sets to a limit $\varphi_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$ which is analytic in the complex disc $|\lambda| < \lambda_o$, continuous in $x_1,\dots,x_k$ and $\alpha_1,\dots,\alpha_k$.[1] ∎

The perturbation series for $\varphi_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$ is most easily expressed in terms of the perturbation series for $\rho_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$, where

$$\rho_\lambda(x_1\alpha_1,\dots,x_k\alpha_k) = \lambda^k e^{\frac{1}{2}\sum_{j=1}^{k}\alpha_j G(o)\alpha_j} \varphi_\lambda(x_1\alpha_1,\dots,x_k\alpha_k) . \qquad (2.16)$$

We shall also give the perturbation series for the truncated correlation functions $\rho^T_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$. The definition of ρ^T_λ is as follows. Let $X = \{x_1\alpha_1,\dots,x_k\alpha_k\}$, then

$$\rho^T_\lambda(X) = \sum_{X=X_1\cup\cdots\cup X_l} (-1)^{l-1}(l-1)!\,\rho_\lambda(X_1)\cdots\rho_\lambda(X_l) ,$$

where the sum is taken over all partitions of X into disjoint subsets $X_1,\dots,X_l$.

By a graph Γ with points $P = \{p_1,\dots,p_n\}$ we shall mean a subset of the set of unordered pairs of differentpoints in P, i.e. $\Gamma \subset \{(p_i p_j) \in P\times P ; i<j\}$. The elements $\gamma \in \Gamma$ are called the lines of the graph Γ. We say that two points $p \in P$ and $q \in P$ are connected in Γ if there exists a sequence of lines in Γ of the form $(p,p_{i_1}),(p_{i_1},p_{i_2}),\dots (p_{i_k},q)$.

If the set of points P of a graph Γ is the union of two disjoint sets P_1 and P_2, $P = P_1 \cup P_2$, and P_1 is called the set of the external points and P_2 the set of the internal points of Γ, then we shall say that Γ is an externally connected graph if every internal point is connected in Γ with an external point. A graph Γ is called connected if any two points of the graph are connected in Γ. With this notations we prove as in Ref[2b] I section 3, lemma 31:

Theorem 2.2

The correlation functions $\rho_\lambda(x_1\alpha_1,\dots,x_n\alpha_n)$ and the truncated correlation functions $\rho_\lambda^T(x_1\alpha_1,\dots,x_n\alpha_n)$ are both given by their convergent powerseries expansions for $|\lambda| < \lambda_o$, and these series are

$$\rho_\lambda(x_1\alpha_1,\dots,x_k\alpha_k) = \sum_{n=o}^{\infty}\frac{\lambda^{n+k}}{n!}\sum_E\int\dots\int\prod_{(i,j)\in E}\{e^{-\alpha_i G(x_i\dots x_j)\alpha_j}-1\}\prod_{i=k+1}^{n}d\nu(\alpha_i)dx_i$$

and

$$\rho_\lambda^T(x_1\alpha_1,\dots,x_k\alpha_k) = \sum_{n=o}^{\infty}\frac{\lambda^{n+k}}{n!}\sum_C\int\dots\int\prod_{(i,j)\in C}\{e^{-\alpha_i G(x_i\dots x_j)\alpha_j}-1\}\prod_{i=k+1}^{n}d\nu(\alpha_i)dx_i\,,$$

where E runs over all externally connected graphs with points $\{1,\dots,k,k+1,\dots,k+n\}$, with $\{1,\dots,k\}$ the external points and $\{k+1,\dots,k+n\}$ the internal points, and C runs over all connected graphs with points $\{1,\dots,k+n\}$. ∎

Remark 1. Theorems 2.1 and 2.2. can also be interpreted as theorems on the correlation functions of a classical continuous (or discrete) gas of particles with inner (continuous or discrete) degrees of freedom (e.g. diatomic molecules). See section 1.

Remark 2.

For the above results on gentle homogeneous perturbations of Gaussian homogeneous random fields we assumed the perturbations to be of the form $\lambda f(s) = \lambda\int e^{is\alpha}d\mu(\alpha)$, with $d\mu(\alpha)$ a complex, bounded measure of compact support on the real line, with $d\mu(-\alpha) = d\overline{\mu(\alpha)}$, [9] and $|\lambda| < \lambda_o$, where λ_o depends on the support and norm of $d\mu(\alpha)$. Actually, following an observation made in a related context by Skripnik [36], [9] we can suppress the condition on the compactness of the support of $d\mu(\alpha)$, provided we change the definition of λ_o. This can be seen by examining the Kirkwood-Salzburg equations ([2a], [36]) for the correlation functions $\rho_\lambda \wedge (x_1\alpha_1,\dots,x_k\alpha_k)$, which can be defined, as well as the Fourier transforms $\varphi_\lambda \wedge (x_1\alpha_1,\dots,x_k\alpha_k)$, exactly as before. The usual proof of the convergence of the

$\rho_\lambda{}^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k)$ as $\Lambda \to R^n(Z^n)$ relies on the observation that the kernel of the Kirkwood-Salzburg equations is independent of Λ and its norm in a suitable Banach space is strictly less that one, provided $|\lambda| < \lambda_o$ ([16],[2a]). To cope with the case where $d\mu(\alpha)$ has unbounded support, it suffices to modify the definition of the Banach space, by taking it to be the closure of the linear vector space of all sequences $\underline{\psi} = \{\psi_n\}$, $n = 1,2,\dots$, with $\psi_n = \psi_n(x_1\alpha_1,\dots,x_n\alpha_n)$ a complex-valued function of $x_1\alpha_1,\dots,x_n\alpha_n$, with norm $\|\psi\|_{\xi,\gamma} = \sup_n \operatorname{ess\,sup}_{\substack{x_1\dots x_n\\ \alpha_1\dots\alpha_n}} \xi^{-n}|\psi_n(x_1\alpha_1,\dots,x_n\alpha_n)| \prod_{i=1}^n e^{\gamma(\alpha_i)}$, where $\gamma(\alpha)$ is a suitable positive function of α. In the case where $d\mu(\alpha)$ has bounded support the choice of Banach space made in [2a] was $\gamma \equiv 0$, $\xi = C^{-1}$. In the present case estimates like the one made in [2a] (see [36]) yield the result that the kernel of the Kirkwood-Salzburg equations has norm less that 1 provided

$$C' = \sup \int e^{\gamma(\alpha')}|e^{-\alpha G(x)\alpha'}-1|d|\nu(\alpha')|dx < \infty$$

(for some choice of $\gamma(\cdot)$) and $|\lambda| < \lambda_o'$, with $\lambda_o' = \xi^{-1}/(\operatorname{ess\,sup}_\alpha e^{\alpha^2\|G(o)\|-\gamma(\alpha)+\xi C'})$. Note that we may choose ξ and $\gamma(\cdot)$ as we like, provided $C' < \infty$. E.g. if we choose $\gamma(\alpha) = A|\alpha|$, with some constant A, then we have $\lambda_o' = C'^{-1}e^{-2B'-1}$, with $B' = \frac{A^2}{8\|G(o)\|^2}(2\|G(o)\|-1)$. Other possible choices can be discussed along the lines of [36].

The functions $\varphi_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$ of Theorem 2.1 have the following cluster property

$$\begin{aligned}&\varphi_\lambda(x_1\alpha_1,\dots,x_k\alpha_k,\ x_{k+1}+a,\alpha_{k+1},\dots,x_{k+l}+a,\alpha_{k+l}) \to \\ &\to \varphi_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)\varphi_\lambda(x_{k+1}\alpha_{k+1},\dots,x_{k+l}\alpha_{k+l})\end{aligned} \tag{2.17}$$

pointwise as a tends to infinity in R^n (respectively Z^n). This follows from the relation (2.16) between the φ_λ and the correlation functions $\rho_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$, and the fact that one can prove cluster properties for the correlation functions ([2a], [16]).

The functions $\varphi_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$, being the limits for $\Lambda \to R^n(Z^n)$ of the Fourier transforms of the finite distributions associated with the probability measure $dP_\lambda{}^\Lambda$, determine a probability measure dP_λ, which is the weak limit of $dP_\lambda{}^\Lambda$ as $\Lambda \to R^n(Z^n)$, such that

$$\varphi_\lambda(x_1\alpha_1,\dots,x_k\alpha_k) = E(e^{i\sum_{j=1}^{k}\alpha_j\xi_\lambda(x_j)}) ,$$

where E is expectation with respect to dP_λ and $\xi_\lambda(x)$ is the random variable given by the measurable function $\xi(x)(\omega)$ considered as measurable function with respect to the measure dP_λ.

The measure dP_λ can be realized e.g. as a measure on the set of tempered distributions $S'(\mathbb{R}^n)$. It is translation invariant because of the translation invariance of the φ_λ.

We shall see later that, under some conditions on the covariance $G_{ij}(x-y)$, the measure is actually on the continuous functions.

The cluster property (2.17) of the φ_λ implies that the measure dP_λ has the cluster property

$$P_\lambda(A \cap B_a) \to P_\lambda(A)\,P_\lambda(B)$$

as $a \to \infty$ in $R^n(Z^n)$, for any measurable sets A,B, where B_a is the set B translated by $a \in \mathbb{R}^n(Z^n)$: $B_a = \{f \mid f(x) = g(x-a) \text{ for some } g \in B\}$.

Remark 3

For $\mu(\alpha) \geq 0$ we can use the connection (2.16) between $\rho_\lambda^\Lambda, \varphi_\lambda$ and $\rho_\lambda, \rho_\lambda^\Lambda$ to relate $\varphi_\lambda^\Lambda, \varphi_\lambda$ and the measures $dP_\lambda^\Lambda, dP_\lambda$ to the well known quantities introduced in classical statistical mechanics for the description of states (e.g. [16], [38]). In particular the ρ_λ determine a Gibssian translation invariant probability measure dG_λ on the space of the denumerable subsets of $R^n(Z^n)$ which have the property that their intersection with any bounded measurable set is finite. dG_λ is the limit of dG_λ^Λ, in the sense that $G_\lambda^\Lambda(A) \to G_\lambda(A)'$ uniformly in A, for any bounded Lebesgue measurable set A in $R^n(Z^n)$, where G_λ^Λ is the Gibbsian measure determined by ρ_λ^Λ:

$$\int_A dG_\lambda^\Lambda = \sum_{k=o}^{|A|} \frac{\lambda^k}{k!} \int_{\Lambda^k} dx_1,\dots,dx_k \int_{R^{km}} d\nu(\alpha_1),\dots,d\nu(\alpha_k)$$

$$\sum_{p=o}^{\infty} (-1)^p \int_{U^p} dx_{k+1},\dots,dx_{k+p} \int_{R^{pm}} d\nu(\alpha_{k+1}),\dots,d\nu(\alpha_{k+p})\rho_\lambda^\Lambda(x_1\alpha_1,\dots,x_{k+p}\alpha_{k+p})$$

and $|A|$ is the number of points in A.

We shall now observe that if we drop the condition $|\lambda| < \lambda_o$ in Theorem 2.1, we can still prove that accumulation points of the ρ_λ^Λ, φ_λ^Λ exist for $\Lambda \to R^n(Z^n)$ but we loose however in general the uniqueness of the limits. In fact from (2.12) we have, for all λ for which $Z_\Lambda \neq 0$ (in particular all $|\lambda| < \lambda_o$ and all $\lambda \geq 0$)

$$\rho_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k) \leq |\lambda|^k e^{\frac{1}{2}\sum_{j=1}^{k}\alpha_j G(o)\alpha_j} |\varphi_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k)|. \qquad (2.18)$$

But by (2.6) we have

$$\varphi_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k) = E(e^{i\sum_{j=1}^{k}\alpha_j\xi_\lambda^\Lambda(x_j)}),$$

where E is expectation with respect to the probability measure dP_λ.

Hence [10]

$$\varphi_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k) \leq 1 \qquad (2.19)$$

and thus from (2.18)

$$\rho_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k) \leq |\lambda|^k e^{\frac{1}{2}\|G(o)\|r^2} \qquad (2.20)$$

Hence the sets of continuous functions $\varphi_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k)$ and $\rho_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k)$ are compact in the dual of bounded continuous functions on $R^{nk}\times R^{mk}$, so that from any sequence Λ_j converging to $R^n(Z^n)$ as $j\to\infty$, in the sense of covering any bounded set, we can extract a subsequence Λ_{j_i} such that

$$\rho_\lambda^{\Lambda_{j_i}}(x_1\alpha_1,\dots,x_k\alpha_k \to \rho_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$$

and $\varphi_\lambda^{\Lambda_{j_i}}(x_1\alpha_1,\dots,x_k\alpha_k)\to\varphi_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$ as $j_i\to\infty$. The convergence is uniform on compacts, for all $\lambda\geq 0$, and the functions ρ_λ satisfy the Kirkwood-Salzburg equations for all $\lambda\geq 0$, due to the continuity of the ρ_λ^Λ as functions of x_i, α_i and the fact that they satisfy the Kirkwood-Salzburg equations.

The ρ_λ, φ_λ are again continuous functions of the $x_1,\dots,x_k$ and $\alpha_1,\dots,\alpha_k$ and are translation invariant.

We state the results in the following

<u>Theorem 2.3.</u>

Under the same assumptions as in Theorem 2.1 we have that, for all $\lambda\geq 0$, the Fourier transforms $\varphi_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k)$ of the joint distributions of the random fields $\xi_\lambda^\Lambda(x_1),\dots,\xi_\lambda^\Lambda(x_k)$ satisfy the uniform bound [10])

$$|\varphi_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k)|\leq 1$$

and the correlation functions $\rho_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k)$ satisfy the bound

$$|\rho_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k)|\leq|\lambda|^k\, e^{\frac{1}{2}\|G(o)\|r^2}\;.$$

For any sequence Λ_j of measurable subsets of $R^n(Z^n)$ which converges to $R^n(Z^n)$ in the sense of covering eventually every bounded subset of $R^n(Z^n)$ we can extract a subsequence Λ_{j_i} such that

$$\varphi_\lambda^{\Lambda_{j_i}}(x_1\alpha_1,\dots,x_k\alpha_k)$$

and

$$\rho_\lambda^{\Lambda_{j_i}}(x_1\alpha_1,\dots,x_k\alpha_k)$$

converge as $\Lambda_{j_i} \to R^n(Z^n)$ uniformly on compacts to limit functions $\varphi_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$, $\rho_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$, which are bounded continuous in the x_i, α_i and translation invariant. ∎

Remark: For a class of classical statistical mechanical systems which correspond to taking $d\nu(\alpha) = \delta_{\frac{1}{\beta^2}}(\alpha)$ and different conditions on $G(x)$, the above result has first been proven by Dobrushin [27] and Ruelle [28].

All we said before on the association of a measure to the set of ρ_λ, φ_λ in the case $|\lambda| < \lambda_o$ can be repeated in the case of ρ_λ, φ_λ given by above Theorem 2.3. Note however, that the quantities φ_λ having been obtained as limits of the φ_λ^Λ through subsequences, to a given set of φ_λ^Λ there can be in general more than one limit quantity φ_λ and therefore the corresponding measure dP_λ is not uniquely determined by the finite distributions dP_λ^Λ. At least in the discrete case of underlying space Z^n there are well known examples [41] which show the non uniqueness of the limits of the ρ_λ^Λ and thus of the φ_λ^Λ.
Of course in general, due to the possible non uniqueness of the limit, the φ_λ do not possess the cluster property and may have singularities in λ, outside the circle $|\lambda| < \lambda_o$.

We would like now to mention some results on the existence of φ_λ (and ρ_λ) that can be derived using a method different from the one of the Kirkwood-Salzburg equations which we have been utilizing above.

We consider the discrete case where the random field $\xi(x)$ is indexed by Z^n. In this case we have

$$\varphi_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k) = Z_\Lambda^{-1}\int e^{i\sum_{j=1}^k \alpha_j\xi(x_j)}\, e^{\lambda\sum_{x\in\Lambda} f(\xi(x))}\, dP_o\,,$$

for any bounded subset $\Lambda \subset Z^n$, where

$$Z_\Lambda = \int e^{\lambda\sum_{x\in\Lambda} f(\xi(x))}\, dP_o \qquad \text{and the } x_j \text{ are points in } Z^n.$$

In this case only the restriction of dP_o to the σ-algebra generated by the $\xi(x_1),\dots,\xi(x_k)$ and the $\xi(x)$ with x in Λ, contributes to the integral.

Suppose Λ consists of l points, $x_{k+1},\dots,x_{k+l}$. Then the expectation $E(e^{i\sum_{j=1}^k \alpha_j\xi_\lambda^\Lambda(x_j)})$ giving $\varphi_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k)$ reduces to an expectation on the finite dimensional measure space
$(R^{n(k+l)}, \prod_{i=1}^{k+l}(2\pi)^{-1/2}e^{-\frac{1}{2}\eta_i^2}d\eta_i)$, so that

$$\varphi_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k) = \Big(\int e^{\lambda\sum_{k+1}^{k+l} f(\eta_j)}\, e^{-\frac{1}{2}\sum_{r,s}\eta_r a_{rs}\eta_s}\, d\eta_{k+1},\dots,d\eta_{k+l}\Big)^{-1}$$
$$\int e^{i\sum_{j=1}^k \alpha_j\eta_j}\, e^{\lambda\sum_{k+1}^{k+l} f(\eta_j)}\, e^{-\frac{1}{2}\sum_{r,s}\eta_r a_{rs}\eta_s}\, d\eta_{k+1},\dots,d\eta_{k+l}\,,$$

where $((a_{rs}))$ is the inverse matrix to the positive definite matrix $G(x_i - x_j)$, $i,j = 1,\dots,k+l$, and is thus symmetric and positive definite. Assume now that the covariance function $G(x)$ is such that $a_{lm} \le 0$ for all $l \neq m$. (Necessary and sufficient for this is ([42], [43]) that $\det((G(x_i-x_j)) \neq 0$, that there exists a non-zero vector with purely non negative components which is mapped by the matrix G into a vector with non negative components and that this property is not shared by any submatrix of G obtained by omitting at least one column). Then the measure

$$\left(\int e^{\lambda \sum\limits_{k+1}^{k+l} f(\eta_j)} e^{-\frac{1}{2}\sum\limits_{r,s} \eta_r a_{rs} \eta_s} d\eta_{k+1},\dots,d\eta_{k+l}\right)^{-1}$$

$$e^{\lambda \sum\limits_{k+1}^{k+l} f(\eta_j)} e^{-\frac{1}{2}\sum\limits_{r,s} \eta_r a_{rs} \eta_s} d\eta_{k+1},\dots,d\eta_{k+l}$$

is a ferromagnetic measure in the sense of [22],[44].

Such measures are known to satisfy general conditions ([22],[44]) sufficient for correlation inequalities to hold. We have in particular the Griffiths inequalities

$$E(\xi_\lambda^\Lambda(x_1),\dots,\xi_\lambda^\Lambda(x_k)) \geq 0$$

$$E((\xi_\lambda^\Lambda(x_1))^{n_1+m_1},\dots,(\xi_\lambda^\Lambda(x_k))^{n_k+m_k}) \geq E((\xi_\lambda^\Lambda(x_1))^{n_1},\dots,(\xi_\lambda^\Lambda(x_k))^{n_k})$$
$$E((\xi_\lambda^\Lambda(x_1))^{m_1},\dots,(\xi_\lambda^\Lambda(x_k))^{m_k})$$

for all integers $n_i, m_i \geq 0$, where E stands for expectation with respect to the measure dP_λ^Λ. Moreover, for any monotone functions F, G of finitely many of the $\xi_\lambda^\Lambda(x_i)$, we have the FKG inequalities

$$E(FG) \geq E(F)E(G).$$

The above inequalities can be used to prove the convergence for $\Lambda \to Z^n$ of the quantities

$S_\lambda^\Lambda(x_1,\dots,x_k) \equiv E(\xi_\lambda^\Lambda(x_1),\dots,\xi_\lambda^\Lambda(x_k))$ and thus of the quantities

$$\varphi_\lambda^\Lambda(x_1\alpha_1,\dots,x_k\alpha_k) = \sum_{l_1,\dots,l_k}^{\infty} \frac{(i\alpha_1)^{l_1}}{l_1!}\cdots\frac{(i\alpha_k)^{l_k}}{l_k!} S_\lambda^\Lambda(x_1,\dots,x_1,x_2,\dots,x_2,\dots,x_k,\dots,x_k),$$

where in the argument of S_λ^Λ the variable x_i occurs l_i times $(i = 1,\dots,k)$. To see this we make the following considerations. Let $\Lambda' \subset \Lambda$. We shall show that $S_\lambda^{\Lambda'}(x_1,\dots,x_k) \leq S_\lambda^\Lambda(x_1,\dots,x_k)$ for $\lambda \leq 0$ ~~by~~ proving that $S_{\lambda,\epsilon g}^\Lambda(x_1,\dots,x_k) \leq S_\lambda^\Lambda(x_1,\dots,x_k)$ for all $\epsilon \geq 0$, where

$$S^{\Lambda}_{\lambda,\epsilon g}(x_1,\dots,x_k) = \frac{E(\xi^{\Lambda}_{\lambda}(x_1),\dots,\xi^{\Lambda}_{\lambda}(x_k)\, e^{\epsilon\lambda\Sigma f(\xi^{\Lambda}_{\lambda}(x))g(x)})}{E(e^{\epsilon\Sigma f(\xi^{\Lambda}_{\lambda}(x))g(x)})} ,$$

where $g(x)$ is an arbitrary non negative function with bounded support in Z^n. To prove $S^{\Lambda}_{\lambda,\epsilon g}(x_1,\dots,x_k) \leq S^{\Lambda}_{\lambda}(x_1,\dots,x_k)$ it is on the other hand enough to show that $S^{\Lambda}_{\lambda,\epsilon g}(x_1,\dots,x_k)$ is a non increasing function of ϵ, for fixed g, since $S^{\Lambda}_{\lambda,0g} = S^{\Lambda}_{\lambda}$. But, using that $S^{\Lambda}_{\lambda,\epsilon g}$ is differentiable with respect to ϵ, we have

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}[S^{\Lambda}_{\lambda,\epsilon g}(x_1,\dots,x_k) - S^{\Lambda}_{\lambda}(x_1,\dots,x_k)] =$$

$$= \lambda E(\xi^{\Lambda}_{\lambda}(x_1),\dots,\xi^{\Lambda}_{\lambda}(x_k)\,\Sigma f(\xi(x))g(x)) -$$

$$- \lambda E(\xi^{\Lambda}_{\lambda}(x_1),\dots,\xi^{\Lambda}_{\lambda}(x_k))\, E(\Sigma f(\xi(x))g(x)).$$

If we assume now that the function $f(\cdot)$ is in the closed linear hull of even polynomials with non negative coefficients, we have then that the difference on the right hand side is non positive for $\lambda \leq 0$ and non negative for $\lambda \geq 0$, as a consequence of the Griffiths inequalities. This proves that, for $\lambda \leq 0$, $S^{\Lambda}_{\lambda,\epsilon g}(x_1,\dots,x_k)$ is a non increasing positive function of ϵ and

$$0 \leq S^{\Lambda}_{\lambda,\epsilon g}(x_1,\dots,x_k) \leq S^{\Lambda}_{\lambda}(x_1,\dots,x_k)$$

for all $\epsilon \geq 0$, $g \geq 0$, g of compact support.

For $\epsilon = 1$ and g equal to the characteristic function of Λ', we obtain thus $S^{\Lambda'}_{\lambda}(x_1,\dots,x_k) \leq S^{\Lambda}_{\lambda}(x_1,\dots,x_k)$ for $\lambda \leq 0$, and the other way for $\lambda > 0$.

Set now, for $\lambda < 0$, $S_{\lambda}(x_1,\dots,x_k) \equiv \inf S^{\Lambda}_{\lambda}(x_1,\dots,x_k)$ and, for $\lambda > 0$, $S_{\lambda}(x_1,\dots,x_k) \equiv \sup S^{\Lambda}_{\lambda}(x_1,\dots,x_k)$, the supremum resp. the infimum being taken over all bounded subsets Λ of Z^n. For any sequence of subsets Λ_j of Z^n which converge to Z^n as $j \to \infty$, in the sense that given any bounded set B there exists N such that for all $j > N$, $\Lambda_j \supset B$, we have then $\lim_{j\to\infty} S_\lambda^{\Lambda_j}(x_1,\ldots,x_k) = S_\lambda(x_1,\ldots,x_k)$. From this convergence and the formula (2.21), we have then $\lim_{j\to\infty} \varphi_\lambda^\Lambda(x_1\alpha_1,\ldots,x_k\alpha_k) \to \varphi_\lambda(x_1\alpha_1,\ldots,x_k\alpha_k)$, with the relation

$$\varphi_\lambda(x_1\alpha_1,\ldots,x_k\alpha_k) = \sum_{l_1,\ldots,l_k} \frac{(i\alpha_1)^{l_1}}{l_1!}\cdots\frac{(i\alpha_k)^{l_k}}{l_k!} S_\lambda^\Lambda(x_1,\ldots,x_1,x_2,\ldots,x_2,\ldots,x_k,\ldots,x_k).$$

We have thus proven that for a class of $f(\cdot)$ in the discrete case of a process indexed by Z^n and for covariances $G(\cdot)$ satisfying a suitable "ferro magnetic condition", we have the convergence of φ_λ^Λ to φ_λ for all $\lambda \leq 0$.

We close this section by a result concerning the support of the measure dP_λ.

Theorem 2.4

Let $\xi(x)$ be a homogeneous Gaussian random field over R^n with mean zero and covariance given by $E(\xi_i(x)\xi_j(x)) = G_{ij}(x-y)$. Assume that the functions $G_{ij}(x)$ satisfy the following three conditions

$$\int |G_{ij}(x)|dx < \infty$$

$$|G_{ij}(x) - G_{ij}(y)| \leq C_1|x-y|^{\alpha_1}, \quad \alpha_1 > 0$$

$$\int_\Lambda |G_{ij}(x_1-y) - G_{ij}(x_2-y)|dy \leq C_2|x_1-x_2|^{\alpha_2}, \quad \alpha_2 > 0,$$

where C_2 and α_2 are independent of Λ, then for all values of λ and for all $f \in C_1(R)$ we have that

$$dP^\Lambda = Z_\Lambda^{-1} e^{\lambda\int_\Lambda f(\xi(x+a_1),\ldots,\xi(x+a_k))dx} dP_0,$$

$$\text{with } Z_\Lambda = \int e^{\lambda\int_\Lambda f\,dx} dP_0,$$

is weakly compact in the Banach space of continuous bounded functions $C(R^n)$, with supremum norm.

In the case $-\lambda_o < \lambda < \lambda_o$ we have that the infinite volume measure dP_λ has support on the bounded continuous functions, so that $\xi_\lambda(x)$ are continuous in x with probability 1.

Proof: This theorem is contained in Theorem 6 of [45]. ∎

3. The Gibbs-state for the anharmonic oscillator

Consider the self adjoint operator

$$H_o = -\tfrac{1}{2}\Delta + \tfrac{1}{2}(x, A^2 x) - \tfrac{1}{2}\operatorname{tr} A \tag{3.1}$$

on the Hilbert space $\mathcal{H} = L_2(R^N)$, where $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ and A is a real symmetric $N \times N$ matrix bounded below by a positive constant, $A \geq cI$, $c > o$, $x \in R^N$ and $(\ ,\ )$ is the natural inner product in R^N.

Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of A. It is well known that H_o has discrete spectrum consisting of the points of the form

$$\sum_{k=1}^{n} \lambda_{i_k} \tag{3.2}$$

and zero. Hence for $\beta > o$, $e^{-\beta H_o}$ is of trace class and we get

$$\operatorname{tr} e^{-\beta H_o} = \sum_{n_1 \geq o, \ldots, n_N \geq o} e^{-\beta \sum_{i=1}^{N} n_i \lambda_i} = \prod_{i=1}^{N} (1 - e^{-\beta \lambda_i})^{-1}, \tag{3.3}$$

so that

$$\operatorname{tr} e^{-\beta H_o} = |1 - e^{-\beta A}|^{-1} \tag{3.4}$$

where $|1 - e^{-\beta A}|$ is the determinant of the matrix $1 - e^{-\beta A}$.

Let $V(x) \geq -b$ be a real measurable function bounded below such that

$$H = H_o + V(x) \tag{3.5}$$

is essentially self adjoint. We say that H is the Hamiltonian for the anharmonic oscillator. From $V \geq -b$ we get $H \geq H_o - b$, which gives us that H has discrete spectrum and together with (3.2) it gives a lower bound for the eigenvalues of H, which is then transformed into an upper bound for the eigenvalues of $e^{-\beta H}$. Therefore we may form the normal state ω_β, on the von Neumann algebra $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$, given by

$$\omega_\beta(B) = (\operatorname{tr} e^{-\beta H})^{-1} \operatorname{tr}(B\, e^{-\beta H}) . \tag{3.6}$$

ω_β is called the Gibbs-state for the anharmonic oscillator.

By the Feynmann-Kac formula we know that the kernel $e^{-\beta H}(x,y)$ of the operator $e^{-\beta H}$ is given by

$$e^{-\beta H}(x,y) = E^\beta_{(x,y)}\left[e^{-\int_o^\beta U(x(\tau))d\tau}\right], \tag{3.7}$$

with $U(x) = \frac{1}{2}(x,A^2x) - \frac{1}{2}\operatorname{tr} A + V(x)$ and $E^\beta_{(x,y)}$ is the conditional expectation with respect to the Brownian motion in R^N given that $x(o) = x$ and $x(\beta) = y$. So that $E^\beta_{(o,o)}$ is the expectation with respect to the normal distribution indexed by the real Hilbert space h of continuous functions $x(\tau)$ from $[o,\beta]$ into R^N, such that $x(o) = x(\beta) = o$ and the norm square

$$\int_o^\beta \left(\frac{dx(\tau)}{d\tau}, \frac{dx(\tau)}{d\tau}\right) d\tau \tag{3.8}$$

is finite.

Consider the Hilbert space $L_2([o,\beta];R^N)$ of L_2-integrable functions from $[o,\beta]$ in R^N, and let $k_{ij}(s,t)$ be the inverse kernel of the self adjoint operator $-\frac{d^2}{d\tau^2}$ with boundary conditions $x(o) = x(\beta) = o$ on $L_2([o,\beta];R^N)$. Then $k_{ij}(s,t) = k(s,t)\delta_{ij}$ and

$$k(s,t) = \begin{cases} \frac{1}{\beta}s(\beta-t) & s \le t \\ \frac{1}{\beta}(\beta-s)t & s \ge t \end{cases} \tag{3.9}$$

The normal distribution indexed by h is the same as the Gaussian process with mean zero and covariance $k_{ij}(s,t)$.

From (3.7) we get that the kernel $e^{-\beta H}(x,y)$ is a continuous function of x and y. It is well known in that case that $\operatorname{tr} e^{-\beta H} = \int e^{-\beta H}(x,x)dx$, which together with (3.7) gives

$$\operatorname{tr} e^{-\beta H} = \int E^{\beta}_{(x,x)}\left[e^{-\int_0^{\beta} U(x(\tau))d\tau}\right]dx . \tag{3.10}$$

It is easy to see that $E^{\beta}_{(x,x)}$ is the expectation with respect to the measure on the continuous periodic functions from $[o,\beta]$ into R^N obtained from the Gaussian process with mean zero and covariance function $k_{ij}(s,t)$ by the transformation $x(\tau) \to x(\tau)+x$.

Since $U(x) = \frac{1}{2}(x,A^2x) - \frac{1}{2}\operatorname{tr} A + V(x)$ we have

$$\begin{aligned}&\int_{R^N} E^{\beta}_{(x,x)}\left[e^{-\int_0^{\beta} U(x(\tau))d\tau}\right]dx \\ &= C_1\int_{R^N} E^{\beta}_{(x,x)}\left[e^{-\frac{1}{2}\int_0^{\beta}(x(\tau),A^2x(\tau))d\tau}\, e^{-\int_0^{\beta} V(x(\tau))d\tau}\right]dx .\end{aligned} \tag{3.11}$$

On the other hand we easily verify that for any real continuous function F defined on the space of continuous periodic functions from $[o,\beta]$ into R^N

$$C_1\int_{R^N} E^{\beta}_{(x,x)}\left[e^{-\frac{1}{2}\int_0^{\beta}(x(\tau),A^2x(\tau))d\tau}\, F\right]dx = C\, E^{\beta}[F] , \tag{3.12}$$

where E^{β} is the expectation with respect to the normal distribution indexed by the real Hilbert space g of continuous periodic functions from $[o,\beta]$ into R^N with norm square

$$\int_0^{\beta}\left[\left(\frac{dx(\tau)}{d\tau}, \frac{dx(\tau)}{d\tau}\right) + (x(\tau),A^2x(\tau))\right]d\tau , \tag{3.13}$$

and C is a normalization constant. By setting $V = 0$ in (3.11) we get $C = \operatorname{tr} e^{-\beta H_o}$. Using (3.4) we have proved the formula

$$\operatorname{tr} e^{-\beta H} = |1 - e^{-\beta A}|^{-1} E^{\beta}\left[e^{-\int_o^{\beta} V(x(\tau))d\tau}\right], \tag{3.14}$$

where E^{β} is the expectation with respect to the normal distribution indexed by the real Hilbert space g, which is the same as the expectation with respect to the homogeneous Gaussian process on the circle S_{β} of length β and covariance function given by

$$E^{\beta}(x_i(o)x_j(t)) = (2A(1-e^{-\beta A}))^{-1}[e^{-tA} + e^{-(\beta-t)A}] \tag{3.15}$$

for $o \leq t \leq \beta$.

For more details and proof of the following theorem, see Ref.[4] section 2.

<u>Theorem 3.1</u>

Let $F_i \in B(\mathcal{H})$ $i = 0,\dots,n-1$ be multiplication operators by bounded continuous functions $F_i(x)$, let $o = s_o \leq \dots \leq s_n = \beta$, and let H be the Hamiltonian for the anharmonic oscillator, then

$$\operatorname{tr}(F_o e^{-s_1 H} F_1 e^{-(s_2-s_1)H} \dots F_{n-1} e^{-(\beta-s_{n-1})H})$$

$$= |1 - e^{-\beta A}|^{-1} E^{\beta}\left[e^{-\int_o^{\beta} V(x(\tau))d\tau} \prod_{i=o}^{n-1} F_i(x(s_i))\right],$$

where $|1 - e^{-\beta A}|$ is the determinant of the matrix $1 - e^{-\beta A}$ and E^{β} is the expectation with respect to the homogeneous Gaussian process on the circle S_{β} of length β with mean zero and covariance function given by

$$E^{\beta}(x_i(o)x_j(t)) = (2A(1 - e^{-\beta A}))^{-1}[e^{-tA} + e^{-(\beta-t)A}]$$

with $o \leq t \leq \beta$.

Let α_t be the C*-automorphism of $B(\mathcal{H})$ defined by

$$\alpha_t(B) = e^{-itH} B\, e^{itH} \tag{3.16}$$

then

$$\mathrm{tr}(B\alpha_t(C)e^{-\beta H}) = \mathrm{tr}(C\, e^{-(\beta-it)H} B\, e^{-itH}) \tag{3.17}$$

is analytic in t in the strip $-\beta < \mathrm{Im}\, t < o$, with boundary values at real t equal to $\mathrm{tr}(B\alpha_t(C)e^{-\beta H})$ and at $t - i\beta$ equal to $\mathrm{tr}(C\alpha_{-t}(B)e^{-\beta H})$.

Moreover

$$\mathrm{tr}(F_o\, e^{-s_1 H} F_1\, e^{-(s_2-s_1)H} \cdots F_{n-1}\, e^{-(\beta-s_{n-1})H}) \tag{3.18}$$

is analytic in the domain $o < \mathrm{Re}\, s_1 < \ldots < \mathrm{Re}\, s_{n-1} < \beta$ with boundary values at $\mathrm{Re}\, s_i = o$ which are continuous and uniformly bounded and for $s_k = it_k$ given by

$$\mathrm{tr}(F_o\alpha_{t_1}(F_1)\alpha_{t_2}(F_2)\ldots\, \alpha_{t_{n-1}}(F_{n-1})e^{-\beta H}). \tag{3.19}$$

<u>Lemma 3.1</u>

Let $t_i \in R$ and F_i be bounded continuous functions on R^N, then $B(\mathcal{H})$ is the smallest strongly closed linear space of operators that contains all operators of the form

$$\alpha_{t_1}(F_1)\alpha_{t_2}(F_2)\ldots\alpha_{t_n}(F_n).$$

For the proof of this lemma and also of the following theorem, see Ref.[4] section 2.

<u>Theorem 3.2</u>

Let B and C be in $B(\mathcal{H})$, then

$$\omega_\beta(B\alpha_t(C)) = \omega_\beta(\alpha_{-t}(B)C)$$

is analytic in the strip $-\beta < \mathrm{Im}\, t < o$, and continuous and uniformly bounded in $-\beta \leq \operatorname{Im} t \leq 0$. The boundary values satisfy the KMS condition

$$\omega_\beta(B\alpha_{t-i\beta}(C)) = \omega_\beta(C\alpha_t(B)) .$$

Moreover any operator $B \in B(\mathcal{H})$ may be approximated strongly by linear combinations of operators of the form $\alpha_{t_1}(F_1) \ldots \alpha_{t_n}(F_n)$, where $F_1, \ldots, F_n$ are multiplication operators by continuous functions $F_1(x) \ldots F_n(x)$. Furthermore $\omega_\beta(F_0\alpha_{t_1}(F_1) \ldots \alpha_{t_n}(F_n))$ is analytic in $0 > \operatorname{Im} t_1 > \ldots > \operatorname{Im} t_n > -\beta$ and its value for $t_k = -is_k$ with $0 \leq s_1 \leq \ldots \leq s_n \leq \beta$ is given by

$$\omega_\beta(F_0\alpha_{-is_1}(F_1) \ldots \alpha_{-is_n}(F_n))$$

$$= \left(E^\beta\left[e^{-\int_0^\beta V(x(\tau))d\tau}\right]\right)^{-1} E^\beta\left[\prod_{i=0}^{n} F_i(x(s_i))e^{-\int_0^\beta V(x(\tau))d\tau}\right]$$

where E^β is the expectation given in theorem 3.1.

4. Statistical Mechanics of a Quantum Lattice System with gentle anharmonic interactions

Let $\Lambda \subset Z^d$ be a bounded subset of the d-dimensional integers lattice. For each $n \in Z^d$ we set $\mathcal{H}_n = L_2(R^m)$, and we define $\mathcal{H}_\Lambda = \underset{n\in\Lambda}{\otimes} \mathcal{H}_n$. $\mathcal{H}_n$ is then the Hilbert space of a quantum mechanical system of m degrees of freedom associated with the lattice point n, and $\mathcal{H}_\Lambda$ is the Hilbert space for the system associated with the set of lattice points Λ. We shall first consider a system of coupled harmonic oscillators. The Hamiltonian $H_o(\Lambda)$ for the system associated with Λ is then given by

$$H_o(\Lambda) = -\tfrac{1}{2} \sum_{n\in\Lambda} \Delta_n + \tfrac{1}{2} \sum_{n,n'\in\Lambda} x_n A(n-n') x_{n'} , \tag{4.1}$$

where $A(n)$ is a positive definite matrix valued function on Z^d, i.e. for any function x_n from Z^d to R^m with compact support, $\Sigma x_n A(n-n')x_{n'} \geq 0$. We shall in fact assume that $A(n)$ satisfies the stronger condition that there are positive numbers b and c such that

$$A(n) = 0 \quad \text{for} \quad |n| > b$$

and

$$\Sigma x_n A(n-n')x_{n'} \geq c \sum_n x_n \cdot x_n \tag{4.2}$$

for some $c > 0$. Δ_n is the Laplacian as self adjoint operator in $L_2(R^m) = \mathcal{H}_n$. Let B be a bounded operator on $\mathcal{H}_{\Lambda_1}$, i.e. $B \in B(\mathcal{H}_{\Lambda_1})$ and $\Lambda_1 \subset \Lambda_2$. Since $\mathcal{H}_{\Lambda_2} = \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2-\Lambda_1}$ we get that $B \to B \otimes 1$ gives a natural embedding $B(\mathcal{H}_{\Lambda_1}) \subset B(\mathcal{H}_{\Lambda_2})$. We shall also consider a system of anharmonic oscillators where the anharmonic term is gentle, in which case the Hamiltonian is of the form

$$H(\Lambda) = H_o(\Lambda) + \lambda \sum_{n\in\Lambda} f(x_{n+a_1}, \dots, x_{n+a_k}), \tag{4.3}$$

where $a_1, \dots, a_k$ are fixed elements in Z^d and $f(x_1, \dots, x_k)$ is a real function of the form

$$f(x_1, \dots, x_k) = \int_{R^{m \cdot k}} e^{i \sum_{j=1}^{k} \alpha_j x_j} d\mu(\alpha_1 \dots \alpha_k),$$

where $d\mu$ is a bounded measure of bounded support in $R^{m \cdot k}$. An interpretation of such systems has been given in Section 1. We shall first study the case $\lambda = 0$. In this case we have from the previous section that for finite Λ the Gibbs-state $\omega_\beta^o(\Lambda)$ at imaginary times is given in terms of a Gaussian process with expectation functional $E_\beta^o(\Lambda)$, and this process is a random field over $\Lambda \times S_\beta$, where S_β is the circle of length β. $E_\beta^o(\Lambda)$ is also the expectation with respect to the normal measure indexed by the Hilbert space of functions on $\Lambda \times S_\beta$ with values in R^m and inner product given by

$$\sum_{n \in \Lambda} \int_o^\beta \left(\frac{dx(\tau,n)}{d\tau}, \frac{dx(\tau,n)}{d\tau}\right) d\tau + \sum_{n,n' \in \Lambda} \int_o^\beta (x(\tau,n), A(n-n')\, x(\tau,n'))\, d\tau \qquad (4.4)$$

Hence the covariance function for the Gaussian process $G_\Lambda(s-t;n,n')$, with s and t in S_β and n and n' in Λ, is the kernel of the inverse of the positive self adjoint operator

$$-\frac{d^2}{dt^2} + A(n-n'), \qquad (4.5)$$

where n and n' are restricted to Λ, on $L_2(\Lambda \times S_\beta; R^m)$. From (4.2) we get that the inverse operator is bounded and in fact that $G_\Lambda(s-t;n,n')$ is a bounded continuous matrix valued function.

As Λ tends to Z^d in the sense that it finally contains all bounded sets in Z^d we get easily that $G_\Lambda(s-t;n,n')$ converges pointwise to a translation invariant function $G(s-t;n-n')$ which is also a bounded continuous function, and in fact it is the kernel of the operator (4.5) defined as a positive self adjoint operator on $L_2(Z^d \times S_\beta ; R^m)$. Again (4.2) gives that $G(s-t\,; n-n')$ is a bounded continuous positive definite matrix valued function. The convergence of G_Λ to G actually follows from the convergence of (4.5) as $\Lambda \to Z^d$. From the convergence of the covariance function $G_\Lambda \to G$ it follows that the corresponding processes or normal measures converge weakly to a limit process, which is the homogeneous Gaussian process on $Z^d \times S_\beta$ with values in R^m and covariance function given by $G(s-t, n-n')$, homogeneneous on Z^d as well as on S_β.

Let us now consider a Hamiltonian of the form

$$H_\Lambda(\Lambda') = H_o(\Lambda') + \lambda \sum_{n\in\Lambda} f(y_n) , \tag{4.6}$$

with $y_n = \{x_{n+a_1}, \dots, x_{n+a_k}\}$, in $\mathcal{H}_{\Lambda'}$, where $\Lambda \subset \Lambda'$. From theorem 3.2 we then have that the Gibbs state $\omega_\beta^\Lambda(\Lambda')$ at imaginary timesis given by

$$\begin{aligned} &\omega_\beta^\Lambda(\Lambda')(\alpha_{-is_1}(F_1) \dots \alpha_{-is_k}(F_k)) \\ &= E_\beta^o(\Lambda')\Big[e^{-\lambda \sum_{n\in\Lambda} \int_o^\beta f(y_n(\tau))d\tau}\Big] E_\beta^o(\Lambda')\Big[\prod_{i=1}^k F_i(x(s_i))\, e^{-\lambda \sum_{n\in\Lambda} \int_o^\beta f(y_n(\tau))d\tau}\Big] , \end{aligned} \tag{4.7}$$

where α_t is the automorphism generated by $H_\Lambda(\Lambda')$ and F_i are bounded continuous functions of the variables $x_n, n \in \Lambda'$, with the notation $x = \{x_n\}_{n\in\Lambda'}$.

By the weak convergence of the Gaussian process with covariance $G_{\Lambda'}$, for which $E_\beta^o(\Lambda')$ is the expectation, to a homogeneous Gaussian process with covariance G and expectation say E_β^o as $\Lambda' \to Z^d$, we get that (4.7) converges as $\Lambda' \to Z^d$ and, denoting the limit by $\omega_\beta^\Lambda(\ \alpha_{-is_1}(F_1) \dots \alpha_{-is_k}(F_k))$, we have by definition

$$\omega_\beta^\Lambda(\alpha_{-is_1}(F_1)\ \ldots\ \alpha_{-is_k}(F_k))$$
$$= E_\beta^o\Big[e^{-\lambda\sum_{n\in\Lambda}\int_o^\beta f(y_n(\tau))d\tau}\Big]^{-1} E_\beta^o\Big[\prod_{i=1}^k F_i(x(s_i))e^{-\lambda\sum_{n\in\Lambda}\int_o^\beta f(y_n(\tau))d\tau}\Big], \tag{4.8}$$

where $F_i(x)$ are functions of $x = \{x_n\}_{n\in Z^d}$ that depend only on a finite number of coordinates x_n, and are bounded continuous functions of these coordinates.

From theorem 3.2 we have that $\omega_\beta^\Lambda(\Lambda')(\alpha_{t_1}(F_1)\ \ldots\ \alpha_{t_n}(F_n))$ are analytic in $0 > \mathrm{Im}\, t_1 > \cdots > \mathrm{Im}\, t_n > -\beta$, uniformly bounded in the closure of this domain and translation invariant in t. Moreover we get that the union of the sets $0 = \mathrm{Im}\, t_1 = \cdots = \mathrm{Im}\, t_k$, $\mathrm{Im}\, t_n = \cdots = \mathrm{Im}\, t_n = -\beta$ as k goes from 0 to n is a distinguished boundary for the domain $0 > \mathrm{Im}\, t_1 > \cdots > \mathrm{Im}\, t_n > -\beta$, so that the absolute value of the function is bounded by its supremum on this boundary. [For an analogue situation, see [33]]. But for real $t_1, \ldots, t_n$ we have, by inspection, that

$$\omega_\beta^\Lambda(\Lambda')(\alpha_{t_1}(F_1)\ \ldots\ \alpha_{t_k}(F_k)\alpha_{t_{k+1}-i\beta}(F_{k+1})\ \ldots\ \alpha_{t_n-i\beta}(F_n))$$
$$= \omega_\beta^\Lambda(\Lambda')(\alpha_{t_{k+1}}(F_{k+1})\ldots\alpha_{t_n}(F_n)\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k)). \tag{4.9}$$

Since $\omega_\beta^\Lambda(\Lambda')$ is a normalized state and α_t is a C*-automorphism for t real we get that the absolute value of (4.9) is bounded by $\prod_{j=1}^n \|F_j\|_\infty$. Hence we get that

$$|\omega_\beta^\Lambda(\Lambda')(\alpha_{t_1}(F_1)\ldots\alpha_{t_n}(F_n))| \leq \prod_{j=1}^n \|F_j\|_\infty \tag{4.10}$$

uniformly in the domain $0 \geq \mathrm{Im}\, t_1 \geq \cdots \geq \mathrm{Im}\, t_n \geq -\beta$. By the convergence at the imaginary points we therefore have the convergence in the domain $0 > \mathrm{Im}\, t_1 > \cdots > \mathrm{Im}\, t_n > -\beta$. This then proves that the function $\omega_\beta^\Lambda(\alpha_{t_1}(F_1)\ldots\alpha_{t_n}(F_n))$, as defined at imaginary points by (4.6), is actually analytic in $0 > \operatorname{Im} t_1 > \dots > \operatorname{Im} t_n > -\beta$ and is bounded in the form

$$|\omega_\beta^\Lambda(\alpha_{t_1}(F_1)\dots\alpha_{t_n}(F_n))| \leq \prod_{j=1}^{n} \|F_j\|_\infty \tag{4.11}$$

in the whole domain.

Next we shall prove that for small values of $|\lambda|$ the limit of ω_β^Λ as $\Lambda \to Z^d$ exists for nice functions $f(x)$.

As before we may assume e.g. that f is of the form

$$f(x_1,\dots,x_k) = \int_{R^m}\dots\int_{R^m} e^{i\sum_{j=1}^{k}\alpha_j x_j}\, d\mu(\alpha_1,\dots,\alpha_k), \tag{4.12}$$

where $\mu(\alpha) = \overline{\mu(-\alpha)}$ is a bounded measure of bounded support in $R^{m\cdot k}$. From (4.6) we have that

$$\omega_\beta^\Lambda(\alpha_{-is_1}(F_1)\cdots\alpha_{-is_k}(F_k)) = \int \prod_{i=1}^{k} F_i(\xi(n_1^i,s_i),\xi(n_2^i,s_i),\dots)dP_\Lambda \tag{4.13}$$

where, in the notations of section 2, $\xi(n,s)$ is the homogeneous Gaussian field over $Z^d \times S_\beta$ with values in R^m and covariance function $G(s-t,n-n')$ and dP_Λ is the measure

$$dP_\Lambda = Z_\Lambda^{-1} e^{\lambda\sum_{n\in\Lambda}\int_0^\beta f(\hat{\xi}(n,s))dx} dP_o\,, \tag{4.14}$$

where $\hat{\xi}(n,s) = \{\xi(n+a_1,s),\dots,\xi(n+a_k,s)\}$ and

$$Z_\Lambda = \int e^{\lambda\sum_{n\in\Lambda}\int_0^\beta f(\hat{\xi}(n,s))ds} dP_o\,, \tag{4.15}$$

dP_o being the normal measure that corresponds to the homogeneous Gaussian process.

From the conditions (4.2) we get that $G(s;n)$ is bounded and falls off exponentially in n, hence there is a $\lambda_o > 0$ such that, for all $|\lambda| < \lambda_o$, we have from section 2 that dP_Λ converges weakly to a limit measure dP. Hence $\omega_\beta^\Lambda(\alpha_{-is_1}(F_1)\dots\alpha_{-is_k}(F_k))$

converges as $\Lambda \to Z^d$ to a function, which we denote $\omega_\beta(\alpha_{-is_1}(F_1)\ldots\alpha_{-is_k}(F_k))$. Since this function is a limit of uniformly bounded analytic functions we get that $\omega_\beta(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))$ is uniformly bounded and analytic in $0 > \operatorname{Im} t_1 > \ldots > \operatorname{Im} t_k > -\beta$ and the bound is

$$|\omega_\beta(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))| \leq \prod_{j=1}^{k} \|F_j\|_\infty . \tag{4.16}$$

Consider now the functions $\omega_\beta^\Lambda(\Lambda')(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))$. We know that they are analytic and uniformly bounded in the same domain, hence we get that

$$\begin{aligned}&\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\omega_\beta^\Lambda(\Lambda')(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))e^{-\frac{1}{2}\sum_{j=1}^{k}(t_j-b_j)^2}dt_1\ldots dt_k\\ &= \int_{\Gamma_1}\ldots\int_{\Gamma_k}\omega_\beta^\Lambda(\Lambda')(\alpha_{z_1}(F_1)\ldots\alpha_{z_k}(F_k)e^{-\frac{1}{2}\sum_{j=1}^{k}(z_j-b_j)^2}dz_1\ldots dz_k ,\end{aligned} \tag{4.17}$$

where Γ_j is given by $\operatorname{Im} z_j = -i\,j\,\epsilon$, $j = 1,\ldots,k$ for any ϵ, $0 < \epsilon < \frac{\beta}{k}$, by the rapid decrease of the integrand. By (4.10) we know that $\omega_\beta^\Lambda(\Lambda')(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))$ is uniformly bounded, and since the functions $e^{-\frac{1}{2}\sum_{j=1}^{k}(t_j-b_j)^2}$ span a dense set in $L_1(R^k)$, we get that convergence of (4.17) as first $\Lambda' \to Z^d$, then $\Lambda \to Z^d$ would imply weak convergence in $L_\infty(R^k)$ of the functions $\omega_\beta^\Lambda(\Lambda')(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))$. By the right hand side of the identity (4.17) however, and the uniform boundedness and the convergence already proved inside $0 > \operatorname{Im} t_1 > \ldots > \operatorname{Im} t_k > -\beta$, we get convergence of (4.17) as first $\Lambda' \to Z^d$ and then $\Lambda \to Z^d$. Moreover, since the identity (4.17) also would hold in the limit, we get that this limit on the real axis, $\omega_\beta(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))$, consists actually of the boundary values of the corresponding analytic function inside the domain.

Since $\omega_\beta(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))$ are limits of $\omega_\beta^\Lambda(\Lambda')(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))$ where $\omega_\beta^\Lambda(\Lambda')$ is a state that is invariant under the corresponding C^*-automorphism α_t (which here actually depends on Λ and Λ'), they will satisfy the positivity condition in an integrated form. Hence we can use the same construction that is used to reconstruct a C^*-algebra and a C^*-automorphism in the case of Wightman functions in quantum field theory (see e.g. [32],[46]). In this way we then get a Hilbert space $\mathcal{H}_\beta$ with a cyclic vector Ω_β for the C^*-algebra generated by operators of the form $\int\alpha_t(F)f(t)dt$, where f is continuous with bounded support, such that

$$\int\ldots\int\omega_\beta(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F))f_1(t_1)\ldots f_k(t_k)dt_1\ldots dt_k$$
$$= (\Omega_\beta, \int\alpha_t(F_1)f_1(t)dt\ldots\int\alpha_t(F_k)f_k(t)dt\,\Omega_\beta), \tag{4.18}$$

where $F_1,\ldots,F_k$ are bounded continuous function of the coordinates at a finite number of lattice points. From (4.10) we have that

$$|\omega_\beta(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))| \le \prod_{j=1}^{k}\|F_j\|_\infty, \tag{4.19}$$

which in fact implies that the operators $\int\alpha_t(F)f(t)dt$ are all bounded operators on $\mathcal{H}_\beta$.

By the translation invariance of $\omega_\beta^\Lambda(\Lambda')$ we get the translation invariance of the function $\omega_\beta(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))$, which gives that Ω_β defines an invariant state for the C^*-automorphisms α_t. Hence α_t is induced by a group of unitary operators U_t on $\mathcal{H}_\beta$. Since $\omega_\beta(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))$ is in L_∞, we get that U_t is weakly continuous, because

$$(\int\alpha_t(F_1)f_1(t)dt\ldots\int\alpha_t(F_n)f_n(t)dt\Omega_\beta, U_s\int\alpha_t(G_1)g_1(t)dt\ldots\int\alpha_t(G_m)g_m(t)dt\,\Omega_\beta)$$

$$= \int\ldots\int\omega_\beta(\alpha_{t_n}(\bar F_n)\ldots\alpha_{t_1}(\bar F_1)\alpha_{s_1}(G_1)\ldots\alpha_{s_m}(G_m)\overline{f_n}(t_n)\ldots\overline{f_1}(t_1)$$
$$g_1(s_1-s)\ldots g_m(s_m-s)dt_1\ldots ds_m$$

converge as $s \to 0$, since g_i is continuous with compact support. The weak continuity of U_t implies the strong continuity of U_t, since U_t is a unitary group. The strong continuity of U_t implies the continuity of the functions $\omega_\beta(\alpha_{t_1}(F_1)\dots\alpha_{t_n}(F_n))$.

Hence we have the following theorem

<u>Theorem 4.1</u>

For the quantum lattice system with finite volume Hamiltonian given by (4.1) and (4.3), there exists a $\lambda_o > 0$ such that for all $|\lambda| < \lambda_o$ the infinite volume limit $\omega_\beta(\alpha_{t_1}(F_1)\dots\alpha_{t_n}(F_n))$ of the Green's functions exists and is analytic in the domain $0 > \mathrm{Im}\, t_1 > \dots > \mathrm{Im}\, t_n > -\beta$ and is continuous and uniformly bounded in the closed domain $0 \geq \mathrm{Im}\, t_1 \geq \dots \geq \mathrm{Im}\, t_n \geq -\beta$.
The uniform bound is given by

$$|\omega_\beta(\alpha_{t_1}(F_1)\dots\alpha_{t_n}(F_n))| \leq \prod_{j=1}^{n} \|F_j\|_\infty .$$

$\omega_\beta(\alpha_{t_1}(F_1)\dots\alpha_{t_n}(F_n))$ is translation invariant with respect to time and lattice translations and at purely imaginary $t_1,\dots,t_n$ it is analytic in λ for λ in the complex disk $|\lambda| < \lambda_o$ and has a convergent linked cluster expansion. In particular the equal time Green's functions are analytic in λ for $|\lambda| < \lambda_o$ and their expansion is the convergent linked cluster expansion. There exists (up to unitary equivalence) a unique Hilbert space $\mathcal{H}_\beta$, with a cyclic vector Ω_β and a strongly continuous unitary group U_t leaving Ω_β invariant, such that

$$\omega_\beta(\alpha_{t_1}(F_1)\dots\alpha_{t_n}(F_n)) = (\Omega_\beta, U_{t_2-t_1} F_1 U_{t_3-t_2}\dots U_{t_n-t_{n-1}} F_n \Omega_\beta) .$$

Moreover Ω_β is invariant under lattice translations and it is the only vector in $\mathcal{H}_\beta$ which has this property. Furthermore, ω_β satisfies the KMS condition.

Proof: Only the two last sentences are not proved already. The invariance of Ω_β under lattice translations follows from the invariance of $\omega_\beta(\alpha_{t_1}(F_1)\ldots\alpha_{t_n}(F_n))$ under lattice translations, which is a consequence of the invariance at the purely imaginary points of the same function, which in turn is implied by the invariance of the measure dP. That Ω_β is the only invariant vector follows from the cluster properties of $\omega_\beta(\alpha_{t_1}(F_1)\ldots\alpha_{t_n}(F_n))$, which in the same way as above is a consequence of the cluster properties of the measure dP which were proved in section 2. The KMS condition follows from the identity

$$\begin{aligned}&\omega_\beta(\alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k)\ \alpha_{t_{k+1}-i\beta}(F_{k+1})\ldots\alpha_{t_n-i\beta}(F_n)) \\ &= \omega_\beta(\alpha_{t_{k+1}}(F_{k+1})\ldots\alpha_{t_n}(F_n)\ \alpha_{t_1}(F_1)\ldots\alpha_{t_k}(F_k))\,,\end{aligned} \tag{4.18}$$

which is a consequence of (4.7). ▨

Remark 1. An explicite value of λ_o can be found in section 2.

Remark 2. This theorem also holds in the case of temperature zero, i.e. $\beta = \infty$, in which case it actually gives the existence of the infinite volume vacuum for the system. The only difference is that the homogeneous Gaussian random field is defined in this case on $R \times Z^d$ and the covariance function $G_\infty(s-t,n-n')$ is the kernel of the inverse of the operator

$$-\frac{d^2}{dt^2} + A(n-n') \tag{4.19}$$

as a self adjoint operator on $L_2(R\times Z^d; R^m)$.

It is easy to see that $G_\infty(s-t,n-n')$ is a bounded function which falls off fast in t and n so that the theorems of section 2 still apply. Moreover, in this case the measure dP has the cluster property also with respect to translations in the t direction, which then gives that Ω_∞ is the only element in the Hilbert space $\mathcal{H}_\infty$ that is invariant under time translations, in this zero temperature case ($\beta = \infty$).

5. Statistical Mechanics of a classical Lattice System with gentle anharmonic interactions.

We shall here consider the classical analogue of the quantum lattice system discussed in the previous section. So let $\Lambda \subset Z^d$ be a bounded subset and we consider a classical potential for the system in Λ of the form

$$V(\Lambda) = \frac{1}{2} \sum_{n,n' \in \Lambda} x_n A(n-n') x_{n'} + \lambda \sum_{n \in \Lambda} f(x_{n+a_1}, \ldots x_{n+a_k}) \tag{5.1}$$

where $x_n \in R^m$, the coordinate space of the system at $n \in Z^d$, and $a_1, \ldots, a_k$ are fixed elements in Z^d, and $f(x_1, \ldots, x_k)$ is again a real function of the form

$$f(x_1, \ldots, x_k) = \int_{R^{m \cdot k}} e^{i \sum_{j=1}^{k} x_j \alpha_j} d\mu(\alpha_1, \ldots, \alpha_k) , \tag{5.2}$$

with $d\mu$ bounded and of bounded support in $R^{m \cdot k}$. Let $F = F(x_{n_1}, \ldots, x_{n_\ell})$ with $n_1, \ldots, n_\ell \in \Lambda$ be a bounded continuous function on the coordinate space. The classical Gibbs-state or Gibbs probability measure for the system in Λ is then given by

$$\rho_\Lambda(F) = Z_\Lambda^{-1} \int \ldots \int F \, e^{-\beta V(\Lambda)} \prod_{n \in \Lambda} dx_n , \tag{5.3}$$

with

$$Z_\Lambda = \int \ldots \int e^{-\beta V(\Lambda)} \prod_{n \in \Lambda} dx_n . \tag{5.4}$$

For $\lambda = 0$ in (5.1) we see that, under condition (4.2) for the harmonic term, the corresponding Gibbs state $\rho_\Lambda^o(F)$ converges to $\rho^o(F)$, where

$$\rho^o(F) = E[F(\xi(n_1), \ldots, \xi(n_\ell))] . \tag{5.5}$$

$\xi(n)$ is the homogeneous Gaussian field on Z^d with values in R^m and with covariance function $\beta^{-1} G(n-n')$, where $G(n-n')$ is the inverse kernel on $L_2(Z^d)$ of the convolution operator given by its kernel $A(n-n')$. By the assumptions (4.2), $G(n-n')$ is bounded and falls off exponentially.

Consider now a potential of the form

$$V_\Lambda(\Lambda') = \tfrac{1}{2} \sum_{n,n'\in\Lambda'} x_n A(n-n') x_{n'} + \lambda \sum_{n\in\Lambda} f(x_{n+a_1},\dots,x_{n+a_k}), \quad (5.6)$$

with $\Lambda \subset \Lambda'$. In this case we find that the corresponding Gibbs state $\rho^\Lambda_{\Lambda'}(F)$ converges as $\Lambda' \to Z^d$ to $\rho^\Lambda(F)$, where

$$\rho^\Lambda(F) = (E[e^{-\lambda\beta \sum_{n\in\Lambda} f(\xi(n+a_1),\dots,\xi(n+a_k))}])^{-1} E[F e^{-\lambda\beta \sum_{n\in\Lambda} f(\xi(n+a_1),\dots,\xi(n+a_k))}]. \quad (5.7)$$

Since $G(n-n')$ satisfies the conditions of theorem 2.1, we find a $\lambda_0 > 0$ such that for $|\lambda| < \lambda_0$ the limit of $\rho^\Lambda(F)$ as $\Lambda \to Z^d$ exists and is analytic in λ for $|\lambda| < \lambda_0$, and the limit state satisfies the cluster property. We have thus the theorem:

Theorem 5.1.

There is a $\lambda_0 > 0$ such that for $|\lambda| < \lambda_0$ the infinite volume classical Gibbs state of a classical system with interaction of the form (5.1), with harmonic term satisfying (4.2) and anharmonic term of the form (5.2), exists and is analytic in λ in the complex disc $|\lambda| < \lambda_0$. Moreover the infinite volume Gibbs state has the cluster property. ∎

It is a fact worth noticing that the classical Gibbs state is the classical limit of the Gibbs state for the corresponding quantum system. That is, if we consider instead of the Hamiltonian (4.3) the Hamiltonian

$$H_\hbar(\Lambda) = -\frac{\hbar^2}{2} \sum_{n\in\Lambda} \Delta_n + \tfrac{1}{2} \sum_{n,n'\in\Lambda} x_n A(n-n') x_{n'} + \lambda \sum_{n\in\Lambda} f(x_{n+a_1},\dots,x_{n+a_k}), \quad (5.8)$$

then the equal times infinite volume Gibbs state for the quantum system with finite volume Hamiltonian given by (5.8), for $|\lambda| < \lambda_o$ converges to the infinite volume Gibbs state for the corresponding classical system as $\hbar \to 0$, where λ_o is the λ_o for the classical system given in theorem 5.1.

To see this, let us write (5.8) in the form

$$H_\hbar(\Lambda) = \hbar^2[-\tfrac{1}{2}\sum_{n\in\Lambda}\Delta_n + \tfrac{1}{2}\sum_{n,n'\in\Lambda} x_n \frac{1}{\hbar^2}A(n-n')x_{n'} + \lambda\beta\sum_{n\in\Lambda}\frac{1}{\hbar^2\beta} f(x_{n+a_1},\dots,x_{n+a_k})] \,. \tag{5.9}$$

From this form of $H_\hbar(\Lambda)$ it follows that the infinite volume Gibbs state at temperature $1/\beta$, for the system with finite volume Hamiltonian given by $H_\hbar(\Lambda)$, is the same as the infinite volume Gibbs state at temperature $1/\hbar^2\beta$ for the system with finite volume Hamiltonian given by

$$-\tfrac{1}{2}\sum_{n\in\Lambda}\Delta_n + \tfrac{1}{2}\sum_{n,n'\in\Lambda} x_n \frac{1}{\hbar^2}A(n-n')x_{n'} + \lambda\beta\sum_{n\in\Lambda}\frac{1}{\hbar^2\beta}f(x_{n+a_1},\dots,x_{n+a_k}) \,. \tag{5.10}$$

By the previous section and theorem 2.2 of section 2 we have the convergent linked cluster expansion for the correlation functions at imaginary time, hence also equal time (= time zero), for the quantum system at temperature $1/\hbar^2\beta$ given by the Hamiltonian (5.10):

$$\rho_\lambda^\hbar(\hat{y}_1\alpha_1,\dots,\hat{y}_k\alpha_k) = \sum_{n=o}^{\infty}\frac{(\beta\lambda)^{n+k}}{n!}\sum_E\int\dots\int \prod_{(i,j)\in E}\{e^{-\alpha_i\hat{G}_\hbar(\hat{y}_i-\hat{y}_j)\alpha_j}-1\}\prod_{i=k+1}^{n}\frac{d\nu(\alpha_i)}{\hbar^2\beta}\,d\hat{y}_i, \tag{5.11}$$

where $\hat{y}_i = \{n_i+a_1,\dots,n_i+a_k;s_i\}$ and $\alpha_i \in R^{m\cdot k}$, $d\nu(\alpha) = e^{-\frac{1}{2}\alpha\hat{G}_\hbar(o)\alpha}d\mu(\alpha)$, with $d\mu(\alpha)$ given in (5.2). The integral is over $S_{\hbar^2\beta} \times Z^d$, and $s_i = 0$ for $1 \le i \le k$. $[\hat{G}_\hbar(\hat{y}_i-\hat{y}_j)]_{p,q} = G_\hbar(n_i-n_j+a_p-a_q)$, where $G_\hbar$ is the kernel of the inverse of the positive self adjoint operator

$$- \frac{d^2}{dt^2} + \frac{1}{\hbar^2} A(n-n') \tag{5.12}$$

on $L_2(S_{\hbar^2\beta} \times Z^d)$. The inverse of (5.12) is actually found as in (3.15) to be

$$(2\frac{B}{\hbar}(1-e^{-\hbar^2\beta\cdot\frac{B}{\hbar}}))^{-1}[e^{-t\frac{B}{\hbar}}+e^{-(\hbar^2\beta-t)\frac{B}{\hbar}}] , \tag{5.13}$$

where $B^2 = A$ and A is the convolution operator with kernel $A(n-n')$. An easy estimate shows that the kernel of (5.13) is continuous in t and for $t = 0$ it is of course given by the kernel of the convolution operator

$$(2\frac{B}{\hbar}(1-e^{-\hbar\beta B}))^{-1}[1+e^{-\hbar\beta B}] . \tag{5.14}$$

As $\hbar \to 0$ we see that (5.14) converges strongly to $\beta^{-1}B^{-2} = \beta^{-1}A^{-1}$. Since these operators are bounded convolution operators on a discrete space Z^d , we have that the kernel converges pointwise, which on Z^d is the same as uniformly on compacts. By the condition (4.2) one verifies easily that the kernel also falls off uniformly exponentially.

The integrations in (5.11) run over $[0,\beta\hbar^2] \times Z^d$, and each integral is multiplied by $(\beta\hbar^2)^{-1}$. This then in fact gives us that each term in the linked cluster expansion converges to the corresponding term in the linked cluster expansion of the correlation functions for the corresponding classical system. By the estimate for the radius of convergence λ_o given in theorem 2.1 we get by an explicite verification that the series (5.11) converges uniformly for $|\lambda| < \lambda_o$, where λ_o is the estimate in theorem 2.1 for the classical system. This then finally gives the convergence of (5.11) towards its classical correspondent as $\hbar \to 0$.

We have now proved the following theorem

Theorem 5.2.

For $|\lambda| < \lambda_0$ the equal time correlation functions for the quantum mechanical system with finite volume Hamiltonian given by (5.8) converge as $h \to 0$ to the corresponding correlation functions for the analogue classical system.

ACKNOWLEDGEMENTS

Instructive discussions with Niels Øvrelid are gratefully aknowledged. The first named author would also like to thank the Institute of Mathematics, University of Oslo, for the warm hospitality as well as the "Norwegian Science Council for Scientific Research and the Humanities (NAVF)" for the financial support.

Footnotes

1) See also e.g. [22b], [1].

2) The reality of f is only required in order that the perturbed measure dP_Λ be real (and in fact a probability measure). Most considerations in this section can be adapted, with adjustments of terminology, to the case of f complex also. The cases where the reality of f is used in an essential way will be pointed out in footnotes.

3) The symmetry condition $\mu(\alpha) = \bar{\mu}(-\alpha)$ is equivalent to the condition that f be real valued, and, as remarked in footnote 2), can be dropped.

4) Such a choice of $d\nu(\alpha)$ gives a measure $d\mu(\alpha)$ which does not satisfy (except for $\beta = 0$) the symmetry condition $\mu(\alpha) = \bar{\mu}(-\alpha)$. See however footnotes 2), 3).

5) Here and in the following we write shortly $\Lambda \to R^n$ with the convention that in the discrete case $\Lambda \to R^n$ should be replaced by $\Lambda \to Z^n$.

6) The condition f real can be dropped, with obvious changes of terminology.

7) Since f is real, $\varphi_\lambda(x_1\alpha_1,\dots,x_k\alpha_k)$ are also positive definite functions of the α_i, $i = 1,\dots,k$, hence in particular almost everywhere differentiable. This argument does not carry over to the case where f is allowed to be complex-valued.

8) See also footnotes 2), 3).

9) This work is related to previous work (see e.g. [37] and references given therein), in which correlation functions of similar systems have been introduced in a discussion of Bogoliubov's S matrix. In [37a] the relation with the canonical system is also discussed.

10) For this estimate we use the assumption that the measure P_Λ is a probability measure, which is a consequence of the symmetry assumption $\mu(\alpha) = \bar{\mu}(-\alpha)$.

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